

ENTROPY CHAOS AND BOSE-EINSTEIN CONDENSATION

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ABSTRACT. We prove the entropy-chaos property for the system of N indistinguishable interacting diffusions rigorously associated with the ground state of N trapped Bose particles in the Gross-Pitaevskii scaling limit of infinite particles. On the path-space we show that the sequence of probability measures of the one-particle interacting diffusion weakly converges to a limit probability measure, uniquely associated with the minimizer of the Gross-Pitaevskii functional.

1. Introduction

Between 2000 to 2002 Lieb, Seiringer and Yngvason solved the problem of giving a physical and mathematical justification of the Gross-Pitaevskii (GP) model for a quantum mechanical particle in a Bose-Einstein condensate ([27],[28], see also [29]). Starting from the N body Hamiltonian, describing the system of N Bose particles in a suitable trapping potential V which interact through a pairwise potential v (see Section 2), they proved that the GP mathematical model can be rigorously obtained from the N body Hamiltonian by performing a suitable limit of infinitely many particles together with a well-defined re-scaling of the interaction potential v . In particular they showed that in this GP scaling limit the one-particle quantum mechanical energy converges to the minimum of the GP functional and that the corresponding interaction energy asymptotically localizes in the points where the other particles are. Moreover they prove the complete or exact Bose-Einstein Condensation ([28]) in terms of the complete factorization of the n -particles reduced density matrix.

More recently, there has been proposed a stochastic description of a Bose-Einstein condensate (the first time in [31]) by using the well-known Nelson map. In 2011 ([34]) it has been shown that under the hypothesis of continuous differentiability of the many body ground state wave function one can define a well-defined one particle stochastic process which, in the GP scaling limit, continuously remains

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outside a time dependent interaction-region with probability one. When the one-particle process is suitably stopped, its convergence in total variation sense can be proved using the relative entropy approach. The convergence is towards a limit diffusion process whose drift is uniquely determined by the minimizer of the Gross-Pitaevskii functional, usually called wave function of the condensate. Successively the phenomenon of the asymptotic localization of relative entropy has been investigated ([35]) as a probabilistic counterpart of the asymptotic localization of the interaction energy. In [44] Kac's chaos was established for the symmetric probability law of the N interacting diffusions system associated to the ground state of the N body Hamiltonian under the GP scaling limit. Since the Nelson map cannot be applied to a non linear Hamiltonian, the problem of correctly individuating the process corresponding to the minimizer of the (non linear) Gross-Pitaevskii functional had to be faced in [2]. Performing a sort of Doob h-transform of the GP Hamiltonian a non linear diffusion generator with a killing rate governed by the wave function of the condensate was derived. The density-dependent killing rate is the probabilistic way to describe the self-interaction suffered from the usual diffusion process with drift of gradient type by the generic other particle which shares the same invariant probability density. With the introduction of a proper one-particle relative entropy an existence theorem for the probability measure associated to the minimizer of the GP functional was proved in [16].

In the present paper we focus on some other probability measures convergence problems related to the GP scaling limit described above. We prove that the entropy chaos property holds for the N interacting diffusions system. This is a stronger chaotic property than Kac's chaos property. The result is obtained, following [22], by using Kac's chaos result plus a regularity condition on the trapping potential V which guarantees a finite moment condition on the corresponding density measures. While Kac's chaos is expressed in terms of weak convergence of all the n -marginal laws toward the n -fold tensor product of the asymptotic probability density, the entropy chaos, which is given by the weak convergence of the one-particle marginal measure plus the convergence of the entropy functionals, allows to prove a total variation convergence result for the same sequence of n -marginal laws. This result concerns the fixed time marginal laws of our N interacting diffusions on the product space \mathbb{R}^{3N} . Since the interaction between the particles asymptotically concentrates on a random region having Lebesgue measure zero (see [34]) but it does not disappear, the convergence problem of the one-particle probability measure on the path space is not trivial and one cannot hope to find a convergence result as strong as the total variation one. In [34], in fact, the latter has been proved only for the stopped version of the one-particle non-markovian diffusion. In this paper we show that on the path space a weak convergence result can be obtained for the one-particle diffusion process by taking advantage of some relevant properties of the corresponding sequences of stopping times. In spite of the strong coupling between the probability measure and the associated stopping time the crucial property is that, for all $t > 0$, the probability that the actual stopping time is larger than t converges to one in the infinite particles limit ([34]). In Section 2 Carlen's class of N interacting diffusions rigorously associated with the ground state of the N body Hamiltonian for N Bose particles is presented.

In Section 3 the main analytical results obtained by Lieb, Seiringer and Yngvason for the specific quantum problem are briefly recalled.

Kac's chaos result for the sequence of N probability laws is described in Section 4. In Section 5 we discuss the concept of entropy chaos and its relation with Kac's chaos. We prove that under an assumption on the form of the trapping potential V the entropy chaos holds for our sequence of N probability laws. Moreover under the hypothesis of convexity of the trapping potential V we establish an inequality of HWI type, i.e. between the relative entropy (H) and the relative Fisher information (I) through the Wasserstein distance (W), which also allows to prove the entropy chaos property. Finally we show that the entropy chaos implies the total variation convergence for the same sequence of probability measures. In Section 6 we establish the weak convergence of the one-particle interacting diffusion on the path space under the GP scaling limit. With this result a probabilistic justification of the GP mathematical model for the condensate is also provided.

2. Nelson-Carlen Diffusions and Bose-Einstein Condensation

Nelson's Stochastic Mechanics is an alternative formulation of Quantum Mechanics which allows to study quantum phenomena using a well determined class of diffusion processes ([36],[37],[5],[6]). See [8] for a more recent review on Stochastic Mechanics.

We will briefly introduce the class of *Nelson* diffusions which are associated to a solution of a Schrödinger equation.

Let the complex-valued function (*wave function*) $\psi(x, t)$ be a solution of the equation:

$$i\partial_t\psi(x, t) = H\psi(x, t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d, \quad (2.1)$$

with $\psi(x, 0) = \psi_0(x)$, corresponding to the Hamiltonian operator:

$$H = -\frac{\hbar^2}{2m}\Delta + V(x),$$

where \hbar denotes the reduced Planck constant, m denotes the mass of a particle, and V is some scalar potential.

Let us set:

$$u(x, t) := \operatorname{Re}\left[\frac{\nabla\psi(x, t)}{\psi(x, t)}\right] \quad (2.2)$$

$$v(x, t) := \operatorname{Im}\left[\frac{\nabla\psi(x, t)}{\psi(x, t)}\right] \quad (2.3)$$

when $\psi(x, t) \neq 0$ and, otherwise, set both $u(x, t)$ and $v(x, t)$ to be equal to zero. Let us put

$$b(x, t) := u(x, t) + v(x, t) \quad (2.4)$$

In a more general approach Carlen ([7]) introduced the following diffusions class, mainly characterized in terms of *proper infinitesimal characteristics* $(\rho_t(x), v_t(x))$ consisting of a time-dependent probability density ρ_t and a time-dependent vector field $v_t(x)$ defined $\rho_t(x)dxdt - a.e.$, so constructed as to have the time-reversal symmetry.

The pairs (ρ_t, v_t) are such that:

$$\int_{\mathbb{R}^d} f(x, T) \rho(x, T) dx - \int_{\mathbb{R}^d} f(x, 0) \rho(x, 0) dx = \int_0^T \int_{\mathbb{R}^d} (v_t \cdot \nabla f)(x, t) dx$$

for all $T \geq 0$ and all $f \in C_0^\infty(\mathbb{R}^{d+1})$

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t)$, with $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$, be the evaluation stochastic process $X_t(\omega) = \omega(t)$, with $\mathcal{F}_t = \sigma(X_s, s \leq t)$ the natural filtration.

Carlen ([5],[6],[7]) proved that if (ρ_t, v_t) is a proper infinitesimal characteristic and if the following *finite energy condition* holds:

$$\int_0^T (\|\nabla \sqrt{\rho_t}\|_{L^2}^2 + \|v_t \sqrt{\rho_t}\|_{L^2}^2) dt < +\infty,$$

with $\|\cdot\|_{L^2}$ the $L^2(\mathbb{R}^d \times \mathbb{R}, dx dt)$ -norm, for all $T \geq 0$ and all $f \in C_0^\infty(\mathbb{R}^{d+1})$, then there exists a unique Borel probability measure \mathbb{P} on Ω such that

- i) $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P})$ is a Markov process;
 - ii) the image of \mathbb{P} under X_t has density $\rho(t, x) := \rho_t(x)$;
 - iii) $W_t := X_t - X_0 - \int_0^t b(X_s, s) ds$
- is a $(\mathbb{P}, \mathcal{F}_t)$ -Brownian Motion.

In this Carlen diffusions class the Nelson diffusions are properly those having the pairs (ρ_t, v_t) of the following form:

$$\rho_t = \psi_t \bar{\psi}_t \quad v_t = \text{Im} \left[\frac{\nabla \psi_t}{\psi_t} \right]$$

($\bar{\psi}_t$ being the conjugate complex function to ψ_t).

The continuity problem for the above Nelson-Carlen map (from solutions of Schrödinger equations to probability measures on the path space given by the corresponding Nelson-Carlen diffusions) is investigated in [13]. For a generalization to the case of Hamiltonian operators with magnetic potential see [40].

From now on we will mainly consider the case where $d = 3$.

We adopt the following notations: capital letters for stochastic processes or, otherwise, we will explicitly specify them, $\hat{Y} = (Y_1, \dots, Y_N)$ to denote arrays in \mathbb{R}^{3N} , $N \in \mathbb{N}$, and bold letters for vectors in \mathbb{R}^3 .

The Hamiltonian introduced to describe the recent experiments ([24], [11]) on BEC is the following N-body Hamiltonian

$$H_N = \sum_{i=1}^N \left(-\frac{\hbar^2}{2m} \Delta_i + V(\mathbf{r}_i) \right) + \sum_{1 \leq i < j \leq N} v(\mathbf{r}_i - \mathbf{r}_j) \quad (2.5)$$

where V is a confining potential, v a pair-wise repulsive interaction potential and $\mathbf{r}_i \in \mathbb{R}^3, i = 1, \dots, N$. It operates on symmetric wave functions Ψ in the complex $L^2(\mathbb{R}^{3N})$ -space in order to satisfy the symmetry permutation prescription for Bose particles.

We consider the mean quantum mechanical energy

$$E[\Psi] = T_\Psi + \Phi_\Psi \quad (2.6)$$

where

$$T_\Psi = \sum_{i=1}^N \int_{\mathbb{R}^{3N}} |\nabla_i \Psi|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N$$

is physically called the *kinetic energy* and

$$\Phi_\Psi = \sum_{i=1}^N \int_{\mathbb{R}^{3N}} V(\mathbf{r}_i) |\Psi|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N + \frac{1}{2} \sum_{i=2}^N \int v(\mathbf{r}_1 - \mathbf{r}_i) |\Psi|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N$$

the *potential energy* associated with Ψ . The variational problem associated to H_N consists in minimizing $E[\Psi]$ with respect to the complex-valued function Ψ in $L^2(\mathbb{R}^{3N})$ subject to the constrain $\|\Psi\|_2 = 1$. If such a minimizing function Ψ_N^0 exists it is called a *ground state*. The corresponding energy $E_0[\Psi_N^0]$ given by

$$E_0[\Psi_N^0] := \inf \{E(\Psi) : \int |\Psi|^2 = 1\}$$

is known as *ground state energy*.

Under suitable assumptions on the potentials V and v one can prove the existence of the ground state Ψ_N^0 for (2.5). Uniqueness of the ground state is to be understood as uniqueness apart from an *overall phase*. For our purposes we need a strictly positive and continuous differentiable ground state. See [41] (Thm.XIII.46 and XIII.47) for the regularities conditions on the potentials V and v implying the strictly positivity and [41] (Thm.XIII.11) for those implying the differentiability of the ground state wave function.

Here we precisely identify the interacting diffusions system rigorously associated to the ground state solution Ψ_N^0 of the Hamiltonian (2.5).

In this case, in fact, the pairs of proper infinitesimal characteristics are of the form

$$(\rho_N, 0) \quad \rho_N := |\Psi_N^0|^2 \quad \sqrt{\rho_N} \in H_1(\mathbb{R}^{3N})$$

with H_1 the Sobolev space of functions with square integrable generalized derivatives.

Introducing the probability space $(\Omega^N, \mathcal{F}^N, \mathcal{F}_t^N, \hat{Y}_t)$, with $\hat{Y}_t(\omega) = \omega(t)$ the evaluation stochastic process, with $\mathcal{F}_t^N = \sigma(Y_s, s \leq t)$ the natural filtration, then by Carlen's Theorem there exists a unique Borel probability measure \mathbb{P}_N such that

- i) $(\Omega^N, \mathcal{F}^N, \mathcal{F}_t^N, \hat{Y}_t, \mathbb{P}_N)$ is a Markov process;
- ii) the image of \mathbb{P}_N under \hat{Y}_t has density $\rho_N(\mathbf{r})$;
- iii) $\hat{W}_t := \hat{Y}_t - \hat{Y}_0 - \int_0^t b_N(\hat{Y}_s) ds$

where

$$b_N(\hat{Y}_t) = \frac{\nabla^{(N)} \Psi_N^0}{\Psi_N^0} = \frac{1}{2} \frac{\nabla^{(N)} \rho_N}{\rho_N}.$$

The stationary probability measure \mathbb{P}_N with density ρ_N can be alternatively defined as the one associated to the Dirichlet form ([19],[20], [32]):

$$\epsilon_{\rho_N}(f, g) := \frac{1}{2} \int_{\mathbb{R}^{3N}} \nabla f(r) \cdot \nabla g(r) \rho_N dr^{3N} \quad f, g \in C_c^\infty(\mathbb{R}^{3N}) \quad (2.7)$$

When Bose-Einstein condensation occurs, the condensate is usually described by the order parameter $\phi_{GP} \in L^2(\mathbb{R}^3)$, also called wave function of the condensate, which is the minimizer of the Gross-Pitaevskii functional

$$E^{GP}[\phi] = \int \left(\frac{\hbar^2}{2m} |\nabla \phi(\mathbf{r})|^2 + V(r) |\phi(\mathbf{r})|^2 + g |\phi(\mathbf{r})|^4 \right) d\mathbf{r} \quad (2.8)$$

under the L^2 -normalization condition

$$\int_{\mathbb{R}^3} |\phi^{GP}(\mathbf{r})|^2 d\mathbf{r} = 1$$

and where $g > 0$ is a parameter depending on the interaction potential v (see also assumption h3) in Section 3 below). Therefore ϕ_{GP} solves the stationary cubic non-linear equation (called Gross-Pitaevskii equation) ([21],[39])

$$-\frac{\hbar^2}{2m} \Delta \phi + V\phi + 2g|\phi|^2\phi = \lambda\phi \quad (2.9)$$

λ , the real-valued Lagrange multiplier of the normalization constraint, is usually called chemical potential. One can prove that ϕ_{GP} is continuously differentiable and strictly positive ([27]).

In [31] the stochastic quantization approach for the system of N interacting Bose particles has been exploited for the first time.

It is proved in [34] that the Stochastic Mechanics of the N -body problem associated to H_N uniquely determines a well defined stochastic process which describes the motion of the single particle in the condensate, in the case of the Gross-Pitaevskii scaling limit as introduced in [27], which allows to prove the existence of an exact Bose-Einstein condensation for the ground state of H^N ([27][28]). For the time-dependent derivation of the Gross-Pitaevskii equation see [1] and [17].

3. Mean energy rescaling according to the GP limit

For simplicity of notations, let us put $\hbar = 2m = 1$.

We consider the mean energy (2.6) expressed in terms of the joint probability density of our $3N$ -dimensional process \hat{Y} as:

$$E_0[\rho_N] = E\left\{ \sum_{i=1}^N [b_i^2(\hat{Y}) + V(Y_i(t))] + \sum_{1 \leq i < j \leq N} v(Y_i(t) - Y_j(t)) \right\}$$

b_i being the drift of the interacting i -th particle, whose position is given by the process Y_i .

Following [27], we assume

h1) $V(|\mathbf{r}_i|)$ is locally bounded, positive and going to infinity when $|\mathbf{r}_i|$ goes to infinity.

h2) v is smooth, compactly supported, non negative, spherically symmetric, with finite positive *scattering length* a ([29] Appendix C).

We perform the following scaling, known as Gross-Pitaevskii (GP) scaling ([27]), writing

h3)

$$v(r) = v_1\left(\frac{r}{a}\right)/a^2$$

$$a = \frac{g}{4\pi N}$$

where v_1 has scattering length equal to 1 and remains fixed while $N \uparrow +\infty$. Moreover $g > 0$ as a consequence of our assumptions on v .

In [27] the following important theorem is proven.

Theorem 3.1. *Under the previous hypothesis h1),h2) h3) one has*

$$\lim_{N \rightarrow \infty} \frac{E_0[\rho_N]}{N} = E^{GP}[\rho_{GP}] \quad (3.1)$$

and

$$\lim_{N \rightarrow \infty} \int \rho_N d\mathbf{r}_2 \cdots \mathbf{r}_N = \rho_{GP} \quad (3.2)$$

where $\rho_{GP} := |\phi_{GP}|^2$, with ϕ_{GP} the minimizer of the Gross-Pitaevskii functional (2.8) and the convergence is in the weak $L^1(\mathbb{R}^3)$ sense.

Remark 3.2. The one-particle marginal density $\rho_N^{(1)}$ converges weakly to ρ_{GP} in the sense that the probability measures $\rho_N^{(1)} d\mathbf{r}$ weakly converge as $N \rightarrow \infty$ towards the probability measure $\rho_{GP} d\mathbf{r}$ on \mathbb{R}^3 .

Using some variational theorems (see [10]) the authors of [27] are also able to uniquely characterize the limit of the single components of the ground state energy $E_0[\rho_N]$.

Theorem 3.3. *(Energy Components) Under the same hypothesis h1),h2),h3) as in Theorem 1, let ϕ_0 denote the solution of the zero-energy scattering equation for v :*

$$-\Delta \phi_0(\mathbf{r}) + \frac{1}{2}v(\mathbf{r})\phi_0(\mathbf{r}) = 0 \quad (3.3)$$

under the boundary condition $\lim_{|\mathbf{r}| \rightarrow +\infty} \phi_0(\mathbf{r}) = 1$ ¹. Putting

$$\hat{s} = \int |\nabla \phi_0|^2 / (4\pi a)$$

with a as in h3), then $\hat{s} \in (0, 1]$ and, recalling that under the assumptions in [27] $|\nabla_1 \Psi_N^0(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2$ exists and it is in $L^1(\mathbb{R}^{3N})$, we have:

$$\begin{aligned} \lim_{N \uparrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^{3N-3}} |\nabla_1 \sqrt{\rho_N}(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N &= \int_{\mathbb{R}^3} |\nabla \sqrt{\rho_{GP}}(\mathbf{r})|^2 d\mathbf{r} + \\ &+ g\hat{s} \int_{\mathbb{R}^3} |\rho_{GP}(\mathbf{r})|^2 d\mathbf{r} \quad (3.4) \end{aligned}$$

¹Setting $\phi_0(\mathbf{r}) = \frac{u(r)}{r}$ with $r = |\mathbf{r}|$ the equation (3.3) is equivalent to $-\Delta u(r) + \frac{1}{2}v(r)u(r) = 0$. The solution of this last equation with $u(0) = 0$, for r larger than the range of v , has the form: $u(r) = \text{const}(r-a)$ with $a > 0$ the scattering length of v . As a consequence $\phi_0(\mathbf{r}) = \text{const}(1 - \frac{a}{r})$. Thus for $r \uparrow +\infty$ we have $\lim_{|\mathbf{r}| \rightarrow +\infty} \phi_0(\mathbf{r}) = 1$.

and, moreover,

$$\lim_{N \uparrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^{3N-3}} V(\mathbf{r}_1) \rho_N(\mathbf{r}_1, \dots, \mathbf{r}_N) d\mathbf{r}_1 \cdots d\mathbf{r}_N = \int_{\mathbb{R}^3} V(\mathbf{r}) \rho_{GP}(\mathbf{r}) d\mathbf{r} \quad (3.5)$$

$$\lim_{N \uparrow \infty} \frac{1}{2} \sum_{j=2}^N \int_{\mathbb{R}^3} \int_{\mathbb{R}^{3N-3}} v(|\mathbf{r}_1 - \mathbf{r}_j|) \rho_N(\mathbf{r}_1, \dots, \mathbf{r}_N) d\mathbf{r}_1 \cdots d\mathbf{r}_N = (1 - \hat{s})g \int_{\mathbb{R}^3} |\rho_{GP}(\mathbf{r})|^2 d\mathbf{r} \quad (3.6)$$

(with $g > 0$ as in (2.8)).

Remark 3.4. As remarked in [34], the above assumptions on Ψ_N^0 are satisfied under regularity assumptions on V and v ([41] Sect. XIII 11).

The next theorem states that the L^2 -distance of the gradient type drifts asymptotically goes to zero if we leave out the neighborhoods of the points where the interaction is localized.

Theorem 3.5. (*Energy Localization*) ([28]). *Defining*

$$F^N(\mathbf{r}_2, \dots, \mathbf{r}_N) := \left(\bigcup_{i=2}^N B^N(\mathbf{r}_i) \right)^c \quad (3.7)$$

where $B^N(\mathbf{r})$ denotes the open ball centered in \mathbf{r} with radius $N^{-\frac{1}{3}-\delta}$ where $0 < \delta \leq \frac{4}{51}$,

$$\lim_{N \uparrow \infty} \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \cdots d\mathbf{r}_N \int_{F^N(\mathbf{r}_2, \dots, \mathbf{r}_N)} \left(\frac{1}{2} \frac{\nabla_1 \rho_N}{\rho_N} - \frac{1}{2} \frac{\nabla_1 \rho_{GP}}{\phi_{GP}} \right)^2 \rho_N d\mathbf{r}_1 = 0 \quad (3.8)$$

4. Kac's chaos in Bose-Einstein Condensation

In this section we put $E = \mathbb{R}^3$ and we consider the symmetric probability law G^N of our N interacting diffusions on the product space E^N .

We describe some results obtained in [44] concerning the asymptotic behavior of our N interacting diffusions (Y_1, Y_2, \dots, Y_N) . The fixed time joint probability density of (Y_1, \dots, Y_N) is given by $\rho_N := |\Psi_N^0|^2$, which is invariant under spatial permutations. In [34] it has been proved that if Ψ_N^0 is the ground state of H_N (as described in section 2) and it is strictly positive and of class C^1 , then the three-dimensional processes $\{Y_i\}_{i=1, \dots, N}$ are equal in law.

We recall the non trivial *chaotic property* introduced by [23]. It properly formalizes the fact that the random variables (Y_1, Y_2, \dots, Y_N) are becoming asymptotically an independent random vector. This is described in terms of the asymptotic factorization of the corresponding symmetric probability laws G_N . One can see this property as the probabilistic counterpart of the *complete Bose-Einstein Condensation* (for the latter result see [28]). Here we consider only the probabilistic setting.

Definition 4.1 ([43]). **(Kac's chaos)**

We say that G_N is *G-Kac's chaotic* (or, equivalently, is chaotic in the Boltzmann's sense) if

$$\forall n \geq 1, \quad G_n^N \rightharpoonup G^{\otimes n}, \quad N \uparrow \infty \quad (4.1)$$

where G_n^N stands for the n -th marginal of G_N and the convergence is the weak convergence of probability measures in the marginal space $E^n = \mathbb{R}^{3n}$.

Following [22], we can reformulate Kac's chaos using the Monge-Kantorovich-Wasserstein (MKW) transportation distance or Wasserstein distance of order 1 between G_n^N and the n -fold tensor product $G^{\otimes n}$.

Definition 4.2. **(MKW distance)**

Given a bounded distance d_E on $E = \mathbb{R}^3$ we introduce the normalized distance d_{E^n} given by

$$\forall X = (x_1, \dots, x_n), Y = (y_1, \dots, y_n) \quad d_{E^n} := \frac{1}{n} \sum_{i=1}^n d_E(x_i, y_i), \quad (4.2)$$

and we define the MKW distance W_1 by

$$W_1(\mu_1, \mu_2) := \inf_{\pi \in \Pi(\mu_1, \mu_2)} \int_{E \times E} d_E(x, y) \pi(dx, dy) \quad (4.3)$$

where $\Pi(\mu_1, \mu_2)$ is the set of probability measures with first marginal μ_1 and second marginal μ_2 .

Definition 4.3. **(Kac's chaos in terms of W_1)**

With the same notations as in Definition 4.1, G_N is *G-Kac's chaotic* if, and only if,

$$\forall n \geq 1, \quad W_1(G_n^N, G^{\otimes n}) \longrightarrow 0 \quad \text{as } N \uparrow \infty \quad (4.4)$$

In [44] the following has been established

Theorem 4.4. **(Kac's chaos for the measure G_N)** *Under the hypothesis h1), h2), h3) (Section 3), the symmetric law G_N is G-Kac's chaotic (in the sense of Definition 4.1).*

With the usual notation

$$\langle \mu, \phi \rangle = \int \phi(x) \mu(dx)$$

for a probability measure μ on E and $\phi \in C_b(E)$, we recall that there is another possible formulation of Kac's chaos in terms of the empirical measures of our interacting diffusions system.

Let us introduced the *empirical measure* on E associated to an E^N -valued random vector $\hat{Y} = (Y_1, Y_2, \dots, Y_N)$:

$$\mu_{\hat{Y}}^N(d\mathbf{r}) := \frac{\sum_{i=1}^N \delta_{Y_i}(d\mathbf{r})}{N}$$

First we recall the definition of chaos in terms of the empirical measure.

Definition 4.5. (Chaos by the empirical measure)

We say that the exchangeable random vector $\hat{Y} = (Y_1, \dots, Y_N)$ is G -chaotic if $\mu_{\hat{Y}}^N$ converges in law to the constant (i.e. deterministic) random variable G .

Following ([43]) one can show that the *chaotic* property in Theorem 3.3 implies a non trivial convergence result for the *empirical measure* (see also [44]).

Proposition 4.6. *If Kac's chaos holds for the probability law G_N , then the empirical measure $\mu_{\hat{Y}}^N(d\mathbf{r}) := \frac{\sum_{i=1}^N \delta_{Y_i}(d\mathbf{r})}{N}$ converges in law as $N \uparrow \infty$ to the constant random variable G . In particular one has for $N \uparrow +\infty$ and $\forall \phi \in C_b(E)$:*

$$E_{\rho_N}[(\langle \mu_{\hat{Y}}^N - G, \phi \rangle)^2] \rightarrow 0$$

It is well-known that the chaotic property according to Definition 4.5 is really an equivalent formulation of Kac's chaos as given in Definition 4.1 (see [43]). For a detailed study of the quantitative dependence between the cited different formulations of Kac's chaos see [22].

We conclude by stressing that the complete Bose-Einstein Condensation proved in [28] implies that Kac's chaos holds for the associated N interacting diffusions system in the GP scaling limit, under the same regularities conditions h1) h2) on the potentials ([44]).

The weak convergence of $\rho_N^{(1)} d\mathbf{r}_1$ to $\rho_{GP} d\mathbf{r}_1$ comes from Theorem 3.1. It is well-known that Kac's chaos is essentially (at least) given by

$$\rho_N^{(2)}(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2 \rightharpoonup \rho_{GP}(\mathbf{r}_1) \rho_{GP}(\mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2 \quad (4.5)$$

This result comes from the proof of the *complete* BEC given in [28]. In particular it may be derived by reducing to the diagonal subspace in the convergence of the 4-particle reduced density matrices (see [44]). The fact that (4.5) is sufficient for having Kac's chaos is a quite classical result (see [43], Proposition 2.2 and [22], Theorem 2.4). The counterpart of this property in the standard analytical framework of BEC is proved in ([28] and [29], Remark after Theorem 7.1) and in ([33], Theorem 7.1.1).

5. Entropy chaos in Bose-Einstein Condensation

In this section we describe the recent concept of entropy chaos, first introduced in [9] and then recently investigated in [22].

First we give the definition of entropy, in terms of a finite moment condition, for probability measures on the product space E^n (see [22]). The entropy, in fact, may not be well-defined for probability measures decreasing too slowly at infinity.

Definition 5.1. (Entropy) The entropy associated with a given measure G_N , admitting a probability density $\rho_N \in L^1(E^N)$ such that $M_k(\rho_N) < \infty$ for some $k > 0$ (M_k denoting the k -th moment), is given by

$$\hat{H}(\rho_N) := \int_{E^N} \rho_N \log(\rho_N) = \int_{E^N} \left(\frac{\rho_N}{H_k} \log\left(\frac{\rho_N}{H_k}\right) - \frac{\rho_N}{H_k} + 1 \right) H_k + \int_{E^N} \rho_N \log H_k \quad (5.1)$$

with $H_k := C_k \exp(-|\mathbf{r}_1|^k - \dots - |\mathbf{r}_N|^k)$, where C_k are normalization constants such that H_k are probability measures on E^N .

Since ρ_N and H_k are probability measures on E^N , the term on the right hand side is well-defined by the fact that the first integral has a non negative integrand and the second one is finite by the assumption of a finite k -th moment. Following [22] we define the normalized entropy functional

$$H(\rho_N) := \frac{1}{N} \hat{H}(\rho_N) \quad (5.2)$$

and we introduce the notion of entropy chaos.

Definition 5.2. (Entropy chaos) A sequence G_N is G -entropy chaotic if

$$G_1^N \rightharpoonup G, \quad H(G_N) \rightarrow H(G) < +\infty, \quad (5.3)$$

where \rightharpoonup stands for weak convergence of measures, as $N \rightarrow +\infty$.

The next theorem (see [22], Theorem 1.4) states that the entropy chaos is a stronger property than Kac's chaos.

Theorem 5.3 ([22]). *If a sequence G_N is G -entropy chaotic, then G_N is G Kac's chaotic.*

Proof. We report a sketch of the proof for completeness (for details see [22]). By definition, the entropy-chaos means that (5.3) holds. We want to prove that for all $n \geq 1$ one has that $G_n^N \rightharpoonup G^{\otimes n}$. Fixing $n \geq 1$, since G_n^N is bounded in E^n , there exists a subsequence $G_n^{N'}$ such that $G_n^{N'} \rightharpoonup F_n$ where F_n is a probability measure on E^n . By the super-additive property of the non normalized entropy \hat{H} (defined according to Definition 5) and by taking the limit using that \hat{H} is lower semi-continuous and bounded by below one has

$$\hat{H}(F_n) \leq \liminf \hat{H}(G_n^{N'}) \leq \liminf \hat{H}(G_n^N) \quad (5.4)$$

Dividing by N and using the convergence of the entropies one obtains

$$H(F_n) \leq \liminf H(G_n^N) = H(G) \quad (5.5)$$

Since the first marginal of F_n is $F_n^1 = G$, G being the limit of G_1^N as $N \uparrow \infty$, and in general, by the properties of the entropy, one has $H(F_n) \geq H(F_n^1)$, then one necessarily obtains $H(F_n) = H(G)$, which implies $F_n = G^{\otimes n}$ a.e.. Since for our subsequence $G_n^{N'}$ we have identified the limit $G^{\otimes n}$, we have proved that the whole sequence G_n^N weakly converges to $G^{\otimes n}$ as $N \uparrow \infty$. \square

Taking advantage of some recent results in [22] we are able to prove that the entropy-chaos holds for our sequence of probability laws G_N under some regularity assumptions on the trapping potential V .

In order to achieve this, it is useful to introduce also the MKW distance of order 2.

Definition 5.4. (MKW distance of order 2)

The MKW distance of order 2, W_2 , is defined as the MKW distance of order 1 with the following choice for the normalized distance d_{E^n}

$$\forall X = (x_1, \dots, x_n), Y = (y_1, \dots, y_n) \quad d_{E^n} := \frac{1}{n} \sum_{i=1}^n |x_i - y_i|^2, \quad (5.6)$$

Later we shall use the relation between W_1 and W_2 given in the following proposition.

Proposition 5.5 ([22]). *Given two symmetric probability measures F^N and G^N on E^N , let us denote for any $k > 0$*

$$\mathcal{M}_k := M_k(F_1^N) + M_k(G_1^N)$$

where $M_k(f)$ denotes the k -th moment of the probability measure f on E .

One has $W_1(F^N, G^N) \leq W_2(F^N, G^N)$. For any $k > 2$, one has

$$W_2(F^N, G^N) \leq 2^{\frac{3}{2}} \mathcal{M}_k^{1/k} W_1(F^N, G^N)^{1/2-1/k}. \quad (5.7)$$

Following [22], we first introduce the *normalized versions* of the relative entropy and the Fisher information between our measures.

Definition 5.6. ((normalized) Relative Entropy) The normalized relative entropy between the measures G_N and $G^{\otimes N}$ is given by

$$H(\rho_N | \rho_{GP}^{\otimes N}) := \frac{1}{N} \int_{E^N} \rho_N \log\left(\frac{\rho_N}{\rho_{GP}^{\otimes N}}\right) \quad (5.8)$$

Remark 5.7. Differently from the entropy functional, the relative entropy does not need any moment regularity assumption because the integrand is a non negative function.

Definition 5.8. (Fisher Information) The Fisher information associated with a given measure G_N is given, when $G_N \in W^{1,1}(E^N)$, by

$$I(\rho_N) := \int_{E^N} \frac{|\nabla \rho_N|^2}{\rho_N} = \int_{E^N} |\nabla \log \rho_N|^2 \rho_N \quad (5.9)$$

In our context the hypothesis of finite energy condition (Section 2) implies that the Fisher Information is well-defined for all N .

Definition 5.9. (normalized) Relative Fisher Information The normalized relative Fisher information between the measures G_N and $G^{\otimes N}$ is given by

$$I(\rho_N | \rho_{GP}^{\otimes N}) := \frac{1}{N} \int_{E^N} |\nabla \log \frac{\rho_N}{\rho_{GP}^{\otimes N}}|^2 \rho_N \quad (5.10)$$

Theorem 5.10. (G_N is G -entropy chaotic)

Under the hypothesis $h1, h2, h3$, and if the trapping potential V is such that $V(\mathbf{r}) \geq \alpha \mathbf{r}^{2+\epsilon} + \beta$ with $\epsilon > 0$, where α and β are constants, the symmetric law G_N is G -entropy chaotic according to Definition 5.2.

Proof. Following ([22], Proposition 3.8), since ρ_N and ρ_{GP} are probability densities in \mathbb{R}^{3N} having finite second moments, we can apply the HWI inequality by Otto-Villani with respect to the Gaussian density $g_\lambda(v) = \frac{1}{(2\pi\lambda)^{3N/2}} \exp(-|v|^2/2\lambda)$ where $v = (\mathbf{r}_1, \dots, \mathbf{r}_N)$ (see [38], Remarks after the proof of Theorem 3):

$$\hat{H}(\rho_N | g_\lambda) \leq \hat{H}(\rho_{GP} | g_\lambda) + W_2(\rho_N, \rho_{GP}) \sqrt{\hat{I}(\rho_N | g_\lambda)} \quad (5.11)$$

Now

$$\hat{H}(\rho_N | g_\lambda) = \hat{H}(\rho_N) - \int \rho_N \log(g_\lambda) = \hat{H}(\rho_N) + \frac{3N}{2} \log(2\pi g_\lambda) + \frac{M_2(\rho_N)}{2\lambda}$$

where M_2 denotes the second moment, and, by definition of relative Fisher information, we have

$$\hat{I}(\rho_N|g_\lambda) = \int \rho_N |\nabla \log(\rho_N) + \frac{v}{\lambda}|^2 = I(\rho_N) + \frac{2}{\lambda} \int v \cdot \nabla \rho_N + \frac{M_2(\rho_N)}{\lambda^2}.$$

By substituting these expressions into (5.11), simplifying the terms containing $\log(g_\lambda)$ and then sending λ to $+\infty$ and dividing the resulting limit by N we obtain the inequality:

$$H(\rho_N) \leq H(\rho_{GP}^{\otimes N}) + W_2(\rho_N, \rho_{GP}^{\otimes N}) \sqrt{I(\rho_N)}.$$

By exchanging the two probability measures ρ_N and $\rho_{GP}^{\otimes N}$, we can recover in the same way:

$$H(\rho_{GP}^{\otimes N}) \leq H(\rho_N) + W_2(\rho_N, \rho_{GP}^{\otimes N}) \sqrt{I(\rho_{GP}^{\otimes N})}$$

and, finally, the inequality

$$|H(\rho_N) - H(\rho_{GP}^{\otimes N})| \leq W_2(\rho_N, \rho_{GP}^{\otimes N}) (\sqrt{I(\rho_N)} + \sqrt{I(\rho_{GP}^{\otimes N})}). \quad (5.12)$$

In order to take advantage of Proposition 5.5, we observe that we must require that the following moment

$$\mathcal{M}_k := M_k(\rho_N^{(1)}) + M_k(\rho_{GP})$$

is finite with $k = 2 + \epsilon$, for some $\epsilon > 0$. This condition is implied by the condition on the confining potential V stated in the theorem.

By applying (5.7) to (5.12) we obtain

$$|H(\rho_N) - H(\rho_{GP}^{\otimes N})| \leq CW_1(\rho_N, \rho_{GP}^{\otimes N})^{1/2-1/k} (\sqrt{I(\rho_N)} + \sqrt{I(\rho_{GP}^{\otimes N})}). \quad (5.13)$$

Since $I(\rho_N)$ is bounded for all N and using the properties of the Fisher information $I(\rho_{GP}^{\otimes N}) = I(\rho_{GP})$ ([22]) and the fact that $I(\rho_{GP})$ is finite, and in addition that the Kac's chaos holds (Theorem 4.4) and the fact that $\rho_N^{(1)}$ weakly converges to ρ_{GP} by Theorem 3.1, we have that the entropy chaos also holds (according to Definition 5.2). \square

Remark 5.11. Since under our assumptions $I(\rho_N)$ is bounded for all N and $I(\rho_{GP})$ is finite, we note that by the relevant inequality (5.12) the entropy chaos would follow from the convergence to zero of the MKW distance of order 2 between the densities ρ_N and $\rho_{GP}^{\otimes N}$. However, to prove this type of convergence between probability measures is in general not a trivial problem (see, e.g. [45], Chapter 6).

We also establish a useful HWI type inequality for the relative entropy between G_N and $G_{GP}^{\otimes N}$.

Theorem 5.12. (HWI inequality) *Under the hypothesis h1), h2), h3), and if the confining potential V is convex and C^∞ one has*

$$H(\rho_N, \rho_{GP}^{\otimes N}) \leq \frac{1}{\sqrt{N}} W_2(\rho_N, \rho_{GP}^{\otimes N}) \sqrt{I(\rho_N, \rho_{GP}^{\otimes N})} \quad (5.14)$$

Proof. Let us introduce W such that $dG^N = \exp(-W)d\mathbf{r}_1 \cdots d\mathbf{r}_N$ on E^N , i.e. $W(\mathbf{r}_1, \dots, \mathbf{r}_N) = -\sum_{i=1}^N \log(\rho_{GP}(\mathbf{r}_i))$. Since $V \in C^\infty(E)$ we have that $\phi_{GP} \in C^\infty$ (see [28]), and, therefore, $W \in C^2(E^N)$. We show now that W is convex. From [27] we know that ϕ_{GP} is log-concave. We report the proof of this fact for completeness. Let us consider the GP equation (2.9). If one is able to prove that the solutions of the equations

$$\partial_t u - \nabla^2 u = 0, \quad \partial_t u + Vu = 0, \quad \partial_t u + 8\pi u^2 = \lambda u \quad (5.15)$$

are log-concave when $u(0, \mathbf{x})$ is positive and log-concave, then applying Trotter formula one has that the solution of equation (2.9) is log-concave. Since the convolution of two log-concave functions is log-concave, the solution of the first equation in (5.15) is log-concave. The solution of the second equation in (5.15) is log-concave due to the convexity of V . The log-concavity of the solution of the third equation in (5.15) is proved in ([30], Theorem 1). Since $W(\mathbf{r}_1, \dots, \mathbf{r}_N) = -\sum_{i=1}^N \log(\rho_{GP}(\mathbf{r}_i))$ and the sum of log-concave functions is also log-concave, it follows that W is convex in E^N .

Since $\rho_N d\mathbf{r}_1 \cdots d\mathbf{r}_N$ is absolutely continuous with respect to $\rho_{GP} d\mathbf{r}_1 \cdots d\mathbf{r}_N$, by ([38], Theorem 3), we have that

$$\hat{H}(\rho_N, \rho_{GP}^{\otimes N}) \leq W_2(\rho_N, \rho_{GP}^{\otimes N}) \sqrt{\hat{I}(\rho_N, \rho_{GP}^{\otimes N})} \quad (5.16)$$

where \hat{H} respectively \hat{I} are the non-normalized relative entropy and respectively relative Fisher information. Dividing by N we finally obtain (5.14). \square

Remark 5.13. We can also prove using Theorem 5.12 that entropy chaos holds. In fact under the assumptions on the confining potential V in Theorem 4.4 plus the assumption of convexity of V , it is sufficient to substitute the inequality in Proposition 5.5 into the HWI inequality (5.14) for obtaining that $H(\rho_N, \rho_{GP}^{\otimes N})$ converges to zero when N goes to infinity.

We have seen that the Kac's chaos property is linked with the weak convergence of the marginal densities. Since under certain regularity conditions on the trapping potential V we have both Kac's chaos and entropy chaos, it is natural to investigate whether the entropy convergence implies a stronger convergence of the marginal densities. The answer is affirmative.

Using some well-known results concerning the convergence of the entropies and the strong L^1 -convergence of the probability densities (see, e.g., [4]), in our case we obtain the following.

Theorem 5.14. *Let G_1^N and G as above. If the entropy-chaos is satisfied (in the sense of Definition 5.2), then G_1^N converges to G in total variation.*

Proof. We first prove that from the convergence of the entropies in (5.3) it follows that

$$H(\rho_N^{(1)}|\gamma) \rightarrow H(\rho_{GP}|\gamma) < +\infty \quad (5.17)$$

where $\gamma(\mathbf{r}) = \frac{1}{(2\pi\lambda)^{3N/2}} \exp(-|\mathbf{r}|^2/2\lambda)$, $\mathbf{r} \in \mathbb{R}^3$, $\lambda > 0$. In fact

$$H(\rho_N^{(1)}) = H(\rho_N^{(1)}|\gamma) + \int \rho_N^{(1)} \log \gamma \quad (5.18)$$

and for $N \uparrow \infty$

$$\int \rho_N^{(1)} \log \gamma \rightarrow \int \rho_{GP} \log \gamma \quad (5.19)$$

By the convergence of the entropies from this (5.17) follows. Now from a well-known result (see, e.g. [4], Lemma 2.5), which is true only on finite measure spaces (see [15] for some counterexamples) , when (5.17) holds and moreover $\rho_N^{(1)} d\mathbf{r} \rightarrow \rho_{GP} d\mathbf{r}$, then as $N \uparrow \infty$

$$\int \left| \frac{\rho_N^{(1)}}{\gamma} - \frac{\rho_{GP}}{\gamma} \right| \gamma d\mathbf{r} \rightarrow 0 \quad (5.20)$$

From the latter result it follows that $\rho_N^{(1)}$ converges to ρ_{GP} strongly in $L^1(d\mathbf{r})$.

Since by using Scheffe's theorem, one can easily prove that in \mathbb{R}^d the total variation distance between two absolutely continuous measures is equal to $\frac{1}{2}$ the L^1 - distance between the two probability densities ² (see, e.g. [14]), we can state that our one-particle marginal probability measure G_1^N converges to G in the sense of total variation convergence of probability measures. \square

6. Existence of a weakly convergent subsequence

In the previous Section 4 and Section 5 we have presented some convergence results for the fixed time marginal density ρ_N .

In the present section we focus on a convergence problem for the one particle probability measure \mathbb{P}_1^N on the path space. First the results contained in [34],[35] and [16] are briefly recalled in order to properly investigate the asymptotic behavior of the *one particle relative entropy*.

We consider the measurable space $(\Omega^N, \mathcal{F}^N)$ where Ω is $C(\mathbb{R}^+ \rightarrow \mathbb{R}^3)$, $N \in \mathbb{N}$ and \mathcal{F} is its Borel sigma-algebra as introduced in Section 2. We denote by $\hat{Y} := (Y_1, \dots, Y_N)$ the coordinate process and by \mathcal{F}_t^N the natural filtration.

Let us introduce a process X^{GP} with invariant density ρ_{GP} and try to compare it with the generic interacting one-particle *non* markovian diffusion $Y_1(t)$.

We assume that X^{GP} is a weak solution of the SDE

$$dX_t^{GP} := u_{GP}(X_t^{GP})dt + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} dW_t \quad (6.1)$$

where,

$$u_{GP} := \frac{1}{2} \frac{\nabla \rho_{GP}}{\rho_{GP}}$$

We denote again by \mathbb{P}_N respectively \mathbb{P}_{GP}^N the measures corresponding to the weak solutions of the $3N$ - dimensional stochastic differential equations

²In fact if \mathbb{Q}_1 and \mathbb{Q}_2 are two absolutely continuous measures in \mathbb{R}^d with densities f_1 and f_2 respectively , then $d_{VT}(\mathbb{Q}_1, \mathbb{Q}_2) := \sup_{A \in \mathcal{B}(\mathbb{R}^d)} |\mathbb{Q}_1(A) - \mathbb{Q}_2(A)| = \sup_{A \in \mathcal{B}(\mathbb{R}^d)} \left| \int_A f_1(\mathbf{r}) d\mathbf{r} - \int_A f_2(\mathbf{r}) d\mathbf{r} \right| = \int_{f_1 > f_2} (f_1(\mathbf{r}) - f_2(\mathbf{r})) d\mathbf{r} = \int_{f_2 > f_1} (f_2(\mathbf{r}) - f_1(\mathbf{r})) d\mathbf{r} = \frac{1}{2} \int |f_1(\mathbf{r}) - f_2(\mathbf{r})| d\mathbf{r}$

$$\hat{Y}_t - \hat{Y}_0 = \int_0^t \hat{b}^N(\hat{Y}_s) ds + \hat{W}_t \quad (6.2)$$

respectively

$$\hat{Y}_t - \hat{Y}_0 = \int_0^t \hat{u}_{GP}(\hat{Y}_s) ds + \hat{W}'_t, \quad (6.3)$$

where

$$\hat{u}_{GP}(\mathbf{r}_1, \dots, \mathbf{r}_N) = (u_{GP}(\mathbf{r}_1), \dots, u_{GP}(\mathbf{r}_N)),$$

\hat{Y}_0 is a random variable with probability density equal to ρ_N , while \hat{W}_t and \hat{W}'_t are $3N$ -dimensional \mathbb{P}_N and \mathbb{P}_{GP}^N standard Brownian motions, respectively.

In this section we use the shorthand notation $\hat{b}_s^N =: \hat{b}^N(\hat{Y}_s)$ and $\hat{u}_s^N =: \hat{u}_{GP}(\hat{Y}_s)$.

Following [34] we compute the relative entropy between the three-dimensional *one-particle* non markovian diffusion Y_1 and X^{GP} .

In order to use Girsanov Theorem, we will assume that u_{GP} is bounded. Then the following finite energy conditions hold:

$$E_{\mathbb{P}_N} \int_0^t \|\hat{b}_s^N\|^2 ds < \infty \quad (6.4)$$

$$E_{\mathbb{P}_N} \int_0^t \|\hat{u}_s^{GP}\|^2 ds < \infty, \quad (6.5)$$

which follow from the fact that Ψ_N^0 is the minimizer of $E^N[\Psi]$, and our hypothesis on u_{GP} . It is well known that these are also *finite entropy conditions* (see, e.g. [18]) which imply that $\forall t \geq 0$

$$\mathbb{P}_N|_{\mathcal{F}_t} \ll \hat{W}|_{\mathcal{F}_t}, \quad \mathbb{P}_{GP}^N|_{\mathcal{F}_t} \ll \hat{W}'|_{\mathcal{F}_t}$$

(where \ll stands for absolute continuity) Then, by Girsanov's theorem, we have, for all $t > 0$,

$$\frac{d\mathbb{P}_N}{d\mathbb{P}_{GP}^N}|_{\mathcal{F}_t} = \exp\left\{-\int_0^t (\hat{b}_s^N - \hat{u}_s^{GP}) \cdot d\hat{W}_s + \frac{1}{2} \int_0^t \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds\right\}, \quad (6.6)$$

where $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^{3N} . The relative entropy restricted to \mathcal{F}_t reads

$$\mathcal{H}(\mathbb{P}_N, \mathbb{P}_{GP}^N)|_{\mathcal{F}_t} =: \mathbb{E}_{\mathbb{P}_N}[\log \frac{d\mathbb{P}_N}{d\mathbb{P}_{GP}^N} |_{\mathcal{F}_t}] = \frac{1}{2} E_{\mathbb{P}_N} \int_0^t \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds \quad (6.7)$$

Since under \mathbb{P}_N the $3N$ -dimensional process \hat{Y} is a solution of (6.2) with invariant probability density ρ_N , we can write, recalling also (6.4) and (6.5),

$$\begin{aligned}
\frac{1}{2}E_{\mathbb{P}^N} \int_0^t \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds &= \\
&= \frac{1}{2} \int_0^t E_{\mathbb{P}^N} \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds = \\
&= \frac{1}{2} t \int_{\mathbb{R}^{3N}} \|\hat{b}^N(\mathbf{r}_1, \dots, \mathbf{r}_N) - \hat{u}^{GP}(\mathbf{r}_1, \dots, \mathbf{r}_N)\|^2 \rho_N d\mathbf{r}_1 \dots d\mathbf{r}_N \quad (6.8)
\end{aligned}$$

so that we get

$$\begin{aligned}
\mathcal{H}(\mathbb{P}_N, \mathbb{P}_{GP}^N)|_{\mathcal{F}_t} &= \\
&= \frac{1}{2} t \int_{\mathbb{R}^{3N}} \sum_{i=1}^N \|b_i^N(\mathbf{r}_1, \dots, \mathbf{r}_N) - u_{GP}(\mathbf{r}_i)\|^2 \rho_N d\mathbf{r}_1 \dots d\mathbf{r}_N = \\
&= \frac{1}{2} N t \int_{\mathbb{R}^{3N}} \|b_1^N(\mathbf{r}_1, \dots, \mathbf{r}_N) - u_{GP}(\mathbf{r}_1)\|^2 \rho_N d\mathbf{r}_1 \dots d\mathbf{r}_N = \\
&= \frac{1}{2} N E_{\mathbb{P}^N} \int_0^t \|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 ds, \quad (6.9)
\end{aligned}$$

where the symmetry of \hat{b}^N and ρ_N has been exploited.

Finally we get the sum of N identical one-particle relative entropies, each of them being defined by

$$\begin{aligned}
\bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP}^N)|_{\mathcal{F}_t} &=: \frac{1}{N} \mathcal{H}(\mathbb{P}_N, \mathbb{P}_{GP}^N)|_{\mathcal{F}_t} = \\
&= \frac{1}{2} E_{\mathbb{P}^N} \int_0^t \|b_1^N(\hat{Y}_s) - u^{GP}(Y_1(s))\|^2 ds \quad (6.10)
\end{aligned}$$

By Theorem 3.3 we deduce that for any $t > 0$ the one particle relative entropy does not go to zero in the scaling limit but it is asymptotically *finite*.

In particular from (3.4) we obtain that:

$$\begin{aligned}
\lim_{N \uparrow +\infty} \bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP}^N)|_{\mathcal{F}_t} &= \\
&= \lim_{N \uparrow +\infty} \frac{1}{2} E_{\mathbb{P}^N} \int_0^t \|b_1^N(\hat{Y}_s) - u^{GP}(Y_1(s))\|^2 ds = \int_0^t g \hat{s} \int_{\mathbb{R}^3} |\rho_{GP}|^2 d\mathbf{r} \quad (6.11)
\end{aligned}$$

where $\hat{s} \in (0, 1]$ is a constant depending on the interaction potential v through the solution of the zero-energy scattering equation (see Theorem 3.3). For every $t \in [0, T]$ with an arbitrary finite T , the right and side of (6.11) is a *finite constant*.

We recall a result, proved in [16], which extends to our case a useful chain-rule for the relative entropy.

Lemma 6.1. *We consider $M = X \times Y$, where X and Y are Polish spaces. Let \mathbb{P} be a measure on M and \mathbb{Q}_1 and \mathbb{Q}_2 measures on X and Y respectively. We denote*

by $\mathbb{Q} = \mathbb{Q}_1 \otimes \mathbb{Q}_2$ the product measure on M of the measures \mathbb{Q}_1 and \mathbb{Q}_2 and we suppose that $\mathbb{P} \ll \mathbb{Q}$. Then we have

$$\mathcal{H}(\mathbb{P}|\mathbb{Q}) \geq \mathcal{H}(\mathbb{P}_1|\mathbb{Q}_1) + \mathcal{H}(\mathbb{P}_2|\mathbb{Q}_2), \quad (6.12)$$

where \mathbb{P}_1 and \mathbb{P}_2 are the marginal probabilities of \mathbb{P} .

We now consider the one-particle marginal \mathbb{P}_N^1 of the symmetric measure \mathbb{P}_N , defined on Ω^N .

The following theorem holds ([16]).

Theorem 6.2 (Marginal Entropy Estimate). *For the relative entropy of the one-particle marginal \mathbb{P}_N^1 versus the measure \mathbb{P}_{GP} one has:*

$$\mathcal{H}(\mathbb{P}_N^1|\mathbb{P}_{GP}) \leq \frac{1}{N} \mathcal{H}(\mathbb{P}_N|\mathbb{P}_{GP}^N) \quad (6.13)$$

Proof. We can use Lemma 6.1. In fact if we decompose $\Omega^N = \Omega \times \Omega^{N-1}$ the hypothesis in Lemma 6.1 are satisfied and we obtain:

$$\mathcal{H}(\mathbb{P}_N|\mathbb{P}_{GP}^N) \geq \mathcal{H}(\mathbb{P}_N^1|\mathbb{P}_{GP}) + \mathcal{H}(\mathbb{P}_N^{N-1}|\mathbb{P}_{GP}^{N-1}), \quad (6.14)$$

where with \mathbb{P}_N^{N-1} we denote the marginal of \mathbb{P}_N with respect to $N-1$ particles and, of course, for the symmetry of \mathbb{P}_N it does not matter which particles we take. Applying again Lemma 6.1 we have:

$$\mathcal{H}(\mathbb{P}_N^{N-1}|\mathbb{P}_{GP}^{N-1}) \geq \mathcal{H}(\mathbb{P}_N^1|\mathbb{P}_{GP}) + \mathcal{H}(\mathbb{P}_N^{N-2}|\mathbb{P}_{GP}^{N-2}), \quad (6.15)$$

So by taking $N-2$ consecutive applications of Lemma 6.1 we obtain the stated result. \square

The next theorem, proved in [16], uses mainly the fact that the relative entropy has the property of the compactness of level sets.

Theorem 6.3 (Existence Theorem). *On the space $(\Omega, \mathcal{F}, \mathbb{P}_{GP})$ there exists a probability measure $\hat{\mathbb{P}}$ such that:*

- i) $\mathbb{P}_{N_j}^1$ weakly converges to $\hat{\mathbb{P}}$ for some subsequence $\mathbb{P}_{N_j}^1$ of \mathbb{P}_N^1 ;
- ii) $\hat{\mathbb{P}}$ is absolutely continuous with respect to \mathbb{P}_{GP} .

7. Uniqueness of the limit probability measure

In this section we prove that $\hat{\mathbb{P}}$ is unique, that is, not only a subsequence of \mathbb{P}_N^1 converges to $\hat{\mathbb{P}}$, but the entire sequence converges to the same limit probability measure. In particular we show that $\hat{\mathbb{P}} = \mathbb{P}_{GP}$.

In order to do this we need some results obtained in [34].

We consider the following time dependent random subset of \mathbb{R}^3

$$D_N(t) := \bigcup_{i=2}^N B^N(Y_i(t)) \quad (7.1)$$

where $B^N(\mathbf{r})$ is again the ball with radius $N^{-\frac{1}{3}-\frac{1}{5}}$ centered in \mathbf{r} (see Lemma 7.3 in [28]), and the stopping time

$$\tau^N := \inf\{t \geq 0 : Y_1(t) \in D_N(t)\} \quad (7.2)$$

7.1. The stopped measures. We investigate some properties of the measure obtained by stopping \mathbb{P}_N with respect to the stopping time τ_N . We start by recalling two relevant properties of τ_N .

The first is that for all $t > 0$ ([34], Proposition 2)

$$\lim_{N \rightarrow +\infty} \mathbb{P}_N(\tau_N \geq t) = 1, \quad (7.3)$$

It is important to stress that τ_N strongly depends on the measure \mathbb{P}_N . We consider, for a fixed value of N and for the one-particle diffusion Y_1 , the random region D_N in \mathbb{R}^3 consisting of the union of $N - 1$ balls, each localized where one of the other $N - 1$ particles is at time t . The joint distribution of the N particles is described by the probability law \mathbb{P}_N .

The second property is as follows. Let us define the stopped measures

$$\tilde{\mathbb{P}}_N = \mathbb{P}_N|_{\mathcal{F}_{\tau_N}},$$

where \mathcal{F}_{τ_N} is the sigma-algebra associated with the stopping time τ_N given by

$$\mathcal{F}_{\tau_N} = \{A \in \mathcal{F}, A \cap \{\tau_N \leq t\} \in \mathcal{F}_t\}$$

We note that $\tilde{\mathbb{P}}_N$ is absolutely continuous with respect to \mathbb{P}_{GP}^N and moreover the following relation holds:

$$\frac{d\tilde{\mathbb{P}}_N}{d\mathbb{P}_{GP}^N} = E_{\mathbb{P}_{GP}^N} \left[\frac{d\mathbb{P}_N}{d\mathbb{P}_{GP}^N} \middle| \mathcal{F}_{\tau_N} \right].$$

In this way we can look at $\tilde{\mathbb{P}}_N$ as a measure defined on the whole sigma algebra \mathcal{F}_T having, by definition, the above density.

Since τ_N represents the stopping time before the one-particle process Y_1 enters in the *interaction random set* $D_N(t)$ generated by the other particles, one can prove that

$$\lim_{N \rightarrow +\infty} \frac{1}{N} H(\tilde{\mathbb{P}}_N || \mathbb{P}_{GP}^N) = 0$$

(see [34], Proposition 3).

Let us now consider $\tilde{\mathbb{P}}_N^1$ as the measure $\tilde{\mathbb{P}}_N$ projected into the sigma-algebra \mathcal{F}^1 generated by the first particle, that is the measure such that

$$\frac{d\tilde{\mathbb{P}}_N^1}{d\mathbb{P}_{GP}^N} = E_{\mathbb{P}_{GP}^N} \left[\frac{d\tilde{\mathbb{P}}_N}{d\mathbb{P}_{GP}^N} \middle| \mathcal{F}^1 \right].$$

Of course in general $\tilde{\mathbb{P}}_N^1 \neq \mathbb{P}_N^1$ and there is no direct connection between the two measures. While $\tilde{\mathbb{P}}_N$ has a clear physical meaning being the measure corresponding to the N particles process stopped according to τ_N , $\tilde{\mathbb{P}}_N^1$ has no physical meaning. In fact when we focus our attention only on the single particle, we cannot know when it will interact with the other particles. This is only possible when we consider all the particles. From the mathematical point of view, this physical fact

corresponds to the non \mathcal{F}^1 -measurability of the stopping time τ_N . By a reasoning similar to the one in Theorem 6.2 we can prove that

$$H(\tilde{\mathbb{P}}_N^1 || \mathbb{P}_{GP}) \leq \frac{1}{N} H(\tilde{\mathbb{P}}_N || \mathbb{P}_{GP}^N) \rightarrow 0.$$

By the well-known Csiszar-Kullback inequality ([12],[25]), we deduce from this that the sequence $\tilde{\mathbb{P}}_N^1$ converges in total variation to \mathbb{P}_{GP} .

We have seen in Section 5 that in \mathbb{R}^d the total variation distance between two absolutely continuous measures is equal to $\frac{1}{2}$ times the L^1 -distance between the corresponding two probability densities. In our path space the following result suffices.

Proposition 7.1. *Given the two absolutely continuous measures $\tilde{\mathbb{P}}_N^1$ and \mathbb{P}_{GP} we have that*

$$d_{TV}(\tilde{\mathbb{P}}_N^1, \mathbb{P}_{GP}) \geq \frac{1}{2} \int \left| \frac{d\tilde{\mathbb{P}}_N^1}{d\mathbb{P}_{GP}} - 1 \right| d\mathbb{P}_{GP} \quad (7.4)$$

Proof. We recall the definition

$$d_{TV}(\tilde{\mathbb{P}}_N^1, \mathbb{P}_{GP}) := \sup_{A \in \mathcal{F}_T^1} |\tilde{\mathbb{P}}_N^1(A) - \mathbb{P}_{GP}(A)| = \sup_{A \in \mathcal{F}_T^1} \left| \int_A \left(\frac{d\tilde{\mathbb{P}}_N^1}{d\mathbb{P}_{GP}} - 1 \right) d\mathbb{P}_{GP} \right| \quad (7.5)$$

By taking

$$A = \left\{ \left(\frac{d\tilde{\mathbb{P}}_N^1}{d\mathbb{P}_{GP}} - 1 \right) \geq 0 \right\}$$

we have that

$$d_{TV}(\tilde{\mathbb{P}}_N^1, \mathbb{P}_{GP}) \geq \int_A \left(\frac{d\tilde{\mathbb{P}}_N^1}{d\mathbb{P}_{GP}} - 1 \right) d\mathbb{P}_{GP} \quad (7.6)$$

and

$$d_{TV}(\tilde{\mathbb{P}}_N^1, \mathbb{P}_{GP}) \geq - \int_{A^c} \left(\frac{d\tilde{\mathbb{P}}_N^1}{d\mathbb{P}_{GP}} - 1 \right) d\mathbb{P}_{GP} \quad (7.7)$$

As a consequence

$$\int \left| \frac{d\tilde{\mathbb{P}}_N^1}{d\mathbb{P}_{GP}} - 1 \right| d\mathbb{P}_{GP} \leq 2d_{TV}(\tilde{\mathbb{P}}_N^1, \mathbb{P}_{GP}) \quad (7.8)$$

□

Since $\tilde{\mathbb{P}}_N^1$ converges in total variation to \mathbb{P}_{GP} , by the previous theorem we obtain that $\frac{d\tilde{\mathbb{P}}_N^1}{d\mathbb{P}_{GP}}$ converges to 1 strongly in $L^1(\mathcal{F}_T^1, \mathbb{P}_{GP})$.

7.2. The unique weak limit. Let us now consider a subsequence of \mathbb{P}_N^1 weakly converging to $\hat{\mathbb{P}}$. For simplicity we denote this subsequence again by \mathbb{P}_N^1 .

Let us recall that, the weak limit $\hat{\mathbb{P}}$ being absolutely continuous with respect to \mathbb{P}_{GP} , for every $A \in \mathcal{F}_T^1, T > 0$, such that $\hat{\mathbb{P}}(\partial A) = \mathbb{P}_{GP}(\partial A) = 0$ one has by definition of weak convergence of measures that

$$\lim_{N \rightarrow +\infty} E_{\mathbb{P}_N^1}[I_A] = E_{\hat{\mathbb{P}}}[I_A].$$

Let us put

$$B_N = \{\tau_N \geq T\}.$$

We recall that $A \cap B_N \in \mathcal{F}_{\tau_N}$. In fact for every $t \in [0, T)$ we have that

$$A \cap B_N \cap \{\tau \leq t\} = \emptyset \in \mathcal{F}_t.$$

Obviously

$$I_A = I_{A \cap B_N} + I_{A \cap B_N^C},$$

where B_N^C is the complement set of B_N . By the properties of the characteristic function I_K of any set K we have that

$$I_{A \cap B_N} \leq I_A \leq I_{A \cap B_N} + I_{B_N^C},$$

from which we obtain that

$$E_{\mathbb{P}_N}[I_{A \cap B_N}] \leq E_{\mathbb{P}_N}[I_A] \leq E_{\mathbb{P}_N}[I_{A \cap B_N}] + \mathbb{P}_N(B_N^C).$$

From the latter relation and the fact that $\mathbb{P}_N(B_N^C) \rightarrow 0$ as $N \rightarrow +\infty$, for every fixed $\epsilon > 0$, we get for N sufficiently big

$$E_{\mathbb{P}_N}[I_{A \cap B_N}] \leq E_{\mathbb{P}_N}[I_A] \leq E_{\mathbb{P}_N}[I_{A \cap B_N}] + \epsilon.$$

By the last relation and $E_{\mathbb{P}_N}[I_A] = E_{\mathbb{P}_N^1}[I_A]$ if we send N to infinity and set $f_+ = \limsup E_{\mathbb{P}_N}[I_{A \cap B_N}]$ and $f_- = \liminf E_{\mathbb{P}_N}[I_{A \cap B_N}]$, we obtain

$$f_+ \leq E_{\mathbb{P}}[I_A] \leq f_- + \epsilon.$$

Finally, $\epsilon > 0$ being arbitrary we have that

$$f = f_+ = f_- = E_{\mathbb{P}}[I_A].$$

We now study what happens to $E_{\mathbb{P}_N}[I_{A \cap B_N}]$ as $N \rightarrow +\infty$.

First of all, since $A \cap B_N \in \mathcal{F}_{\tau_N}$ we have that

$$E_{\mathbb{P}_N}[I_{A \cap B_N}] = E_{\tilde{\mathbb{P}}_N}[I_{A \cap B_N}].$$

Recalling that $I_{A \cap B_N} = I_A I_{B_N}$, consequently we obtain

$$\begin{aligned} E_{\tilde{\mathbb{P}}_N}[I_{A \cap B_N}] &= E_{\tilde{\mathbb{P}}_N}[I_A I_{B_N}] \\ &= E_{\tilde{\mathbb{P}}_N^1}[E_{\tilde{\mathbb{P}}_N}[I_A I_{B_N} | \mathcal{F}_T^1]] \\ &= E_{\tilde{\mathbb{P}}_N^1}[I_A E_{\tilde{\mathbb{P}}_N}[I_{B_N} | \mathcal{F}_T^1]] \\ &= E_{\mathbb{P}_{GP}} \left[I_A \frac{d\tilde{\mathbb{P}}_N^1}{d\mathbb{P}_{GP}} E_{\tilde{\mathbb{P}}_N}[I_{B_N} | \mathcal{F}_T^1] \right]. \end{aligned}$$

Let us set

$$H_N := E_{\tilde{\mathbb{P}}_N}[I_{B_N} | \mathcal{F}^1].$$

By the properties of the conditional expectation and of the characteristic function we have that $0 \leq H_N \leq 1$. Therefore H_N is a sequence which is bounded in norm in $L^\infty(\mathcal{F}_T^1, \mathbb{P}_{GP})$.

Since $L^1(\mathcal{F}_T^1, \mathbb{P}_{GP})$ is separable, \mathcal{F}_T^1 being a sub- σ algebra of the essentially separable σ algebra \mathcal{F}_T , then by considering the dual pair $\langle L^1, L^\infty \rangle$, there exists a subsequence H_{N_j} which is weak* convergent to \tilde{H} in L^∞ by a well-known result on weak* convergence.

We have previously proved that $\frac{d\tilde{\mathbb{P}}_N^1}{d\mathbb{P}_{GP}} \rightarrow 1$ strongly in L^1 , and so, I_A being a bounded function, we have that

$$I_A \frac{d\tilde{\mathbb{P}}_N^1}{d\mathbb{P}_{GP}} \rightarrow I_A,$$

strongly in L^1 .

By the properties of weak and strong convergence we obtain that

$$E_{\mathbb{P}_{GP}} \left[I_A \frac{d\tilde{\mathbb{P}}_{N_j}^1}{d\mathbb{P}_{GP}} H_{N_j} \right] \rightarrow E_{\mathbb{P}_{GP}} [I_A \tilde{H}],$$

i.e.

$$E_{\mathbb{P}_{GP}} [I_A \tilde{H}] = f = E_{\hat{\mathbb{P}}} [I_A]. \quad (7.9)$$

Since this happens for an arbitrary set $A \in \mathcal{F}_T^1$ which is a \mathbb{P}_{GP} -continuity set (i.e. $\mathbb{P}_{GP}(\partial A) = 0$) then the equality (7.9) is true for any set in \mathcal{F}_T^1 . In fact, let us consider the class \mathcal{C}_b of the finite intersections of open balls whose boundaries have null \mathbb{P}_{GP} measure. This class is a π system (i.e. it is closed with respect to finite intersections) because

$$\partial(B \cap C) \subseteq (\partial B) \cup (\partial C)$$

The equality (7.9) is evidently true for any set in \mathcal{C}_b . Since Ω is separable, then $\sigma(\mathcal{C}_b) = \mathcal{F}_T^1$, i.e. the \mathcal{C}_b generates the Borel σ -algebra \mathcal{F}_T^1 . In fact we know that by separability the finite intersections of open balls generates \mathcal{F}_T^1 because each open set is a countable union of open balls. On the other hands, one can prove that the open balls having boundary with null \mathbb{P}_{GP} measure are dense in the following sense. Given an arbitrary open ball $B(x, \epsilon)$, $\epsilon > 0$, there exists an $r \in (0, \epsilon)$ such that the boundary of $B(x, r)$ has \mathbb{P}_{GP} measure zero because for different r these boundaries have empty intersection. Therefore we have that

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}_{GP}} = \tilde{H}.$$

From this fact we deduce that not only the subsequence H_{N_j} converges to \tilde{H} , but also the entire sequence H_N converges to \tilde{H} weakly* in L^∞ . Moreover since $0 \leq H_N \leq 1$ by the properties of weak* convergence we must have $0 \leq \tilde{H} \leq 1$.

Of course from $E_{\mathbb{P}_{GP}}[\tilde{H}] = 1$, it necessarily follows that $\tilde{H} = 1$. This implies that

$$\hat{\mathbb{P}} = \mathbb{P}_{GP}.$$

Since every weakly convergent subsequence of \mathbb{P}_N^1 converges to \mathbb{P}_{GP} , we can deduce that the entire sequence \mathbb{P}_N^1 weakly converges to \mathbb{P}_{GP} .

We have proved the following:

Theorem 7.2. (*A weak convergence result*)

Under the hypothesis h1), h2), and h3), the one-particle marginal measure \mathbb{P}_N^1 weakly converges to \mathbb{P}_{GP} , the weak solution to (6.1), where ϕ_{GP} is the unique minimizer of the GP functional (2.8).

Remark 7.3. For the proof of our convergence result we did not have at disposal the well-known Lemma 11.1.1 by Stroock-Varadhan ([42]). The reason for this is that we can not decouple the sequence of stopping times τ_N from the sequence of probability measures \mathbb{P}_N because they are intrinsically coupled, in our case.

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